Rutgers University: Algebra Written Qualifying Exam

August 2016: Problem 3 Solution

Exercise. Let I be an ideal in a principal ideal domain R. Show if $I \neq R$, then

$$\bigcap_{n=1}^{\infty} I^n = (0).$$

(Here I^n is the ideal generated by all products $x_1 \dots x_n$ such that $x_i \in I$ for all $i = 1, \dots, n$)

Solution.

Suppose **for contradiction,** $x \in \bigcap_{n=1}^{\infty} I^n$ and $x \neq 0$.
Since I is an ideal and R is a principal ideal domain,

$$I = \langle a \rangle = Ra = \{ra : r \in R\}.$$

$$\begin{aligned}
x \in I^n &\implies x \in Ra^n && \forall n \\
&\implies \exists b_1, \dots, b_n \in R \text{ s.t. } x = b_n a^n && \forall n \\
&\implies b_n a^n = b_{n+1} a^{n+1} \\
&\implies b_n = b_{n+1} a \\
Rb_1 \subseteq Rb_2 &\subseteq \cdots \subseteq Rb_n
\end{aligned}$$

R is a PID $\implies$ is **noetherian**:

$$Rb_n = Rb_{n+1} \text{ for some } n$$

$\implies b_n = r_1 b_{n+1}$ and $r_2 b_n = b_{n+1}$ for some $r_1, r_2 \in R$
$\implies a$ must be a unit $\implies I = Ra = R$, a contradiction.